

# Inverse moment problem for elementary co-adjoint orbits

Leonid Faybusovich <sup>\*</sup>  
 Department of Mathematics  
 University of Notre Dame  
 Notre Dame, IN 46556  
 Leonid.Faybusovich.1@nd.edu

Michael Gekhtman  
 Department of Mathematics  
 University of Notre Dame  
 Notre Dame, IN 46556  
 Michael.Gekhtman.1@nd.edu

February 8, 2008

## Abstract

We give a solution to the inverse moment problem for a certain class of Hessenberg and symmetric matrices related to integrable lattices of Toda type.

## 1 Introduction

This paper continues our previous work [8, 9], where we dealt with the family of integrable Hamiltonian systems in  $\mathbb{R}^{2n}$  parametrized by index sets  $I = \{i_1, \dots, i_k : 1 < i_1 < \dots < i_k = n\}$  and generated by Hamiltonians

$$H_I(Q, P) = \frac{1}{2} \sum_{i=1}^n P_i^2 + \sum_{1 \leq i < n; i \neq i_1, \dots, i_{k-1}}^n P_i e^{Q_{i+1} - Q_i} + \sum_{j=1}^{k-1} e^{Q_{i_j+1} - Q_{i_j}}. \quad (1.1)$$

This family contains (after an appropriate coordinate changes) such important integrable systems as the standard and relativistic Toda lattices, Volterra lattice and lattices of the Ablowitz- Ladik hierarchy.

In [8], we argued that the full Kostant-Toda flows on Hessenberg matrices provide a convenient framework to study systems generated by (1.1). Namely, each of these systems possesses a Lax representation with a Lax operator given by an  $n \times n$  upper Hessenberg matrix  $X_I = X_I(Q, P)$  that belongs to a certain  $(2n - 2)$ -dimensional co-adjoint orbit of the upper triangular group. This orbit is determined by  $I$  and the value of  $Tr(X_I) = \sum_{i=1}^n P_i$ . Furthermore,  $Tr(X_I^2) = H_I$  and Hamilton equations of motion are equivalent to the Toda flow  $\dot{X}_I = [X_I, (X_I)_{\leq 0}]$ , where  $(A)_{\leq 0}$  denotes the lower triangular part of a matrix  $A$ . The orbit contains a dense open set of elements that admit a factorization into a product of elementary bi-diagonal matrices. Written in terms of parameters of this factorization, equations of motion become a particular case of the constrained KP lattice studied in [13].

Each of the systems described above can be linearized via the Moser map  $X \rightarrow m(\lambda, X) = ((\lambda \mathbf{1} - X)^{-1} e_1, e_1) = \sum_{j=0}^{\infty} \frac{s_j(X)}{\lambda^{j+1}}$  where  $e_1 = (1, 0, \dots, 0)$  and  $s_j(X) = (X^j e_1, e_1)$  (see [11, 3, 6, 5]). Moreover, as we have shown in [9] the Moser map is very useful in establishing a multi-Hamiltonian structure of these systems. However, explicit formulas for the inverse of the Moser map seem

---

<sup>\*</sup>Research partially supported by the NSF grant DMS98-03191

to be known only in two cases. If  $X_I$  is tri-diagonal ( $I = \{2, \dots, n\}$ ) they give a solution in terms of Hankel determinants to the classical finite-dimensional moment problem (see, e.g. [1]). In the "opposite" case that corresponds to the relativistic Toda lattice ( $I = \{n\}$ ) the solution to the inverse problem is given in terms of Toeplitz determinants constructed from the moments  $s_j(X_I), j = -n, \dots, n$  (see [10]).

The main purpose of this paper is to give a solution to the inverse problem for any  $I$ . Explicit formulas (given, once again, in terms of Toeplitz determinants) are obtained in sect. 3.

As in the Hessenberg case, the inverse problem for elementary co-adjoint orbits in the symmetric case was previously studied for  $I = \{2, \dots, n\}$  and  $I = \{n\}$ . The latter case corresponds to peakons solutions of the shallow water equation and was recently comprehensively studied in [2]. We treat the symmetric case in sect. 4 for an arbitrary  $I$ .

We would like to express our gratitude to Yu. Suris for valuable comments and suggestions.

## 2 Elementary Orbits in the Hessenberg Case

Everywhere below we denote by  $e_{jk}$  an elementary matrix  $(\delta_\alpha^i \delta_\beta^k)_{\alpha, \beta=1}^n$  and by  $e_j$  a column vector  $(\delta_\alpha^j)_{\alpha=1}^n$  of the standard basis in  $\mathbb{R}^n$ .

Denote by  $J$  an  $n \times n$  matrix with 1s on the first sub-diagonal and 0s everywhere else. Let  $\mathfrak{b}_+, \mathfrak{n}_+, \mathfrak{b}_-, \mathfrak{n}_-$  be, resp., algebras of upper triangular, strictly upper triangular, lower triangular and strictly lower triangular matrices. Denote by  $\mathcal{H}$  the set  $J + \mathfrak{b}_+$  of upper Hessenberg matrices.

For any matrix  $A$  we write its decomposition into a sum of lower triangular and strictly upper triangular matrices as

$$A = A_- + A_0 + A_+$$

and define  $A_{\geq 0} = A_0 + A_+, A_{\leq 0} = A_0 + A_-, A_{sym} = A_+ + A_0 + A_+^T$ .

A linear Poisson structure on  $\mathcal{H}$  is obtained as a pull-back of the Kirillov-Kostant structure on  $\mathfrak{b}_-^*$ , the dual of  $\mathfrak{b}_-$ , if one identifies  $\mathfrak{b}_-^*$  and  $\mathcal{H}$  via the trace form. A Poisson bracket of two functions  $f_1, f_2$  on  $\mathcal{H}$  then reads

$$\{f_1, f_2\}(X) = \langle X, [(\nabla f_1(X))_{\leq 0}, (\nabla f_2(X))_{\leq 0}] \rangle, \quad (2.1)$$

where we denote by  $\langle X, Y \rangle$  the trace form  $\text{Trace}(XY)$  and gradients are computed w.r.t. this form. Symplectic leaves of the bracket (2.1) are orbits of the coadjoint action of the group  $\mathbf{B}_-$  of lower triangular invertible matrices:

$$\mathfrak{O}_{X_0} = \{J + (\text{Ad}_n X_0)_{\geq 0} : n \in \mathbf{B}_-\}. \quad (2.2)$$

Following [8], consider a family of orbits whose members are parameterized by increasing sequences of natural numbers  $I = \{i_1, \dots, i_k : 1 < i_1 < \dots < i_k = n\}$ . To each sequence  $I$  there corresponds a 1-parameter family of  $2(n-1)$ -dimensional coadjoint orbits

$$M_I = \cup_{\nu \in \mathbb{R}} \mathfrak{O}_{X_I + \nu \mathbf{1}} \subset \mathcal{H}, \quad (2.3)$$

where

$$X_I = e_{1i_1} + \sum_{j=1}^{k-1} e_{i_j i_{j+1}} + J. \quad (2.4)$$

We call orbits  $\mathfrak{D}_{X_I+\nu\mathbf{1}}$  *elementary*.

The set  $M'_I$  of elements of the form

$$X = (J + D)(\mathbf{1} - C_k)^{-1}(\mathbf{1} - C_{k-1})^{-1} \cdots (\mathbf{1} - C_1)^{-1}, \quad (2.5)$$

where  $D = \text{diag}(d_1, \dots, d_n)$

$$C_j = \sum_{\alpha=i_{j-1}}^{i_j-1} c_\alpha e_{\alpha, \alpha+1}, \quad (2.6)$$

is dense in  $M_I$ .

The following formulae express entries  $x_{lm}$  ( $l < m$ ) of  $X$  in terms of  $c_i, d_i$  (see [8]) :

$$x_{lm} = d_l u_{lm} + u_{l-1, m} = \begin{cases} (d_l + c_{l-1})c_l \cdots c_{m-1}, & i_{j-1} < l < m \leq i_j \\ d_{i_{j-1}}c_{i_{j-1}} \cdots c_{m-1}, & i_{j-1} = l < m \leq i_j \\ d_l + c_{l-1}, & l = m \\ 0, & \text{otherwise} \end{cases} \quad (2.7)$$

(Here  $c_0 = 0$ .)

Define a sequence  $\epsilon_1, \dots, \epsilon_n$  by setting

$$\epsilon_i = \begin{cases} 0 & \text{if } i = i_j \text{ for some } 0 < j \leq k \\ 1 & \text{otherwise} \end{cases}. \quad (2.8)$$

Then  $X$  can also be written as<sup>1</sup>

$$X = (J + D)(\mathbf{1} + U_1)(\mathbf{1} - U_2)^{-1}, \quad (2.9)$$

where

$$U_1 = \sum_{\alpha=1}^{n-1} (1 - \epsilon_\alpha) c_\alpha e_{\alpha, \alpha+1}, U_2 = \sum_{\alpha=1}^{n-1} \epsilon_\alpha c_\alpha e_{\alpha, \alpha+1} \quad (2.10)$$

In what follows, we will also use a sequence of integers  $(\nu_i)_{i=1}^n$  defined by

$$\nu_i = i(1 - \epsilon_i) - \sum_{\beta=1}^{i-1} \epsilon_\beta \quad (2.11)$$

It is easy to check that

$$\nu_i = \begin{cases} j & \text{if } i = i_j \text{ for some } 0 < j \leq k \\ -\sum_{\beta=1}^{i-1} \epsilon_\beta & \text{otherwise} \end{cases}. \quad (2.12)$$

It follows from (2.12) that

- (i) sequences  $(\epsilon_i)$  and  $(\nu_i)$  uniquely determine each other.
- (ii) If we define, for every  $i \in \{1, \dots, n\}$ , a set  $N_i = \{\nu_\alpha : \alpha = 1, \dots, i\}$  and a number  $k_i = \max\{j : i_j \leq i\} = i - \sum_{\beta=1}^i \epsilon_\beta$ , then

$$N_i = \{k_i - i + 1, \dots, k_i - 1, k_i\} = \{1 - \sum_{\beta=1}^i \epsilon_\beta, \dots, i - \sum_{\beta=1}^i \epsilon_\beta\} \quad (2.13)$$

---

<sup>1</sup>This was suggested to us by Yu. Suris [14]

### 3 Solution of the Inverse Problem

Assume that all  $d_i \neq 0$  and then define the moment sequence  $S = (s_i, i \in \mathbb{Z})$  of  $X$ :

$$s_i = s_i(X) = e_1^T X^i e_1. \quad (3.1)$$

Our goal is to express coefficients  $c_i, d_i$  in terms of  $S$ . In fact, only a segment  $s_{k+1-n}, \dots, s_{n+k}$  will be needed.

Let  $p(\lambda) = (p_0(\lambda) := 1, p_1(\lambda), \dots, p_{n-1}(\lambda))$  be a solution of the truncated eigenvalue problem

$$(p(\lambda)X)_i = \sum_{\alpha=1}^i x_{\alpha i} p_{\alpha-1}(\lambda) + p_i(\lambda) = \lambda p_{i-1}(\lambda), \quad i = 1, \dots, n-1 \quad (3.2)$$

or, equivalently,

$$(p(\lambda)(J + D)(\mathbf{1} + U_1))_i = \lambda((\mathbf{1} - U_2)p)_i(\lambda), \quad i = 1, \dots, n-1. \quad (3.3)$$

Clearly, such solution exists for every  $\lambda$  and is uniquely defined. Moreover, each  $p_i(\lambda)$  is a monic polynomial of degree  $i$ . We can re-write (3.3) as a 3-term recursion for polynomials  $p_i(\lambda)$ :

$$p_{i+1}(\lambda) + b_{i+1}p_i(\lambda) + (1 - \epsilon_i)a_i p_{i-1}(\lambda) = \lambda(p_i(\lambda) - \epsilon_i c_i p_{i-1}(\lambda)), \quad i = 0, \dots, n-1, \quad (3.4)$$

where

$$b_i = d_i + (1 - \epsilon_{i-1})c_{i-1}, \quad a_i = d_i c_i. \quad (3.5)$$

**Lemma 3.1** *For any upper Hessenberg matrix  $X$ , polynomials  $p_i(\lambda)$  defined by (3.2) satisfy*

$$p_i(X)e_1 = e_{i+1}, \quad i = 0, \dots, n-1. \quad (3.6)$$

**Proof.** Since

$$Xe_i = \sum_{\alpha=1}^i x_{\alpha i} e_\alpha + e_{i+1}$$

and, by (3.2),

$$Xp_{i-1}(X) = \sum_{\alpha=1}^i x_{\alpha i} p_{\alpha-1}(X)e_1 + p_i(X)e_1,$$

one concludes that sequences of vectors  $(e_i)_{i=1}^n$  and  $(p_{i-1}(X)e_1)_{i=1}^n$  are defined by the same recurrence relations.  $\square$

**Lemma 3.2** *For  $i = 1, \dots, n-1$ , a subspace  $\mathfrak{L}_i$  generated by vectors  $(e_\alpha^T)_{\alpha=1}^i$  coincides with a subspace generated by vectors  $(e_1^T X^{\nu_\alpha})_{\alpha=1}^i$ , where  $\nu_\alpha$  are defined in (2.11). In other words, for some constants  $\gamma_i \neq 0$*

$$\gamma_i e_i^T = \begin{cases} e_1^T X^j & \text{if } i = i_j \\ e_1^T X^{-\sum_{\beta=1}^{i-1} \epsilon_\beta} & \text{otherwise} \end{cases} \pmod{\mathfrak{L}_{i-1}}. \quad (3.7)$$

Moreover, for  $i = 2, \dots, n$ ,

$$\gamma_i = e_1^T X^{i(1-\epsilon_i) - \sum_{\beta=1}^{i-1} \epsilon_\beta} e_i = (-1)^{\epsilon_i(i-1)} c_1 d_1 \frac{\prod_{\beta=2}^{i-1} d_\beta^{1-\epsilon_\beta} c_\beta}{(d_1 \cdots d_i)^{\epsilon_i}}. \quad (3.8)$$

**Proof.** For any  $X$  given by (2.5), (2.9), consider an upper triangular matrix

$$V = D(\mathbf{1} - C_k)^{-1}(\mathbf{1} - C_{k-1})^{-1} \cdots (\mathbf{1} - C_1)^{-1} = D(\mathbf{1} + U_1)(\mathbf{1} - U_2)^{-1}.$$

(If  $D$  is invertible, then  $V$  is an upper triangular factor in the Gauss factorization of  $X$ .)

By (2.6),  $e_l^T C_j = 0$  for  $l \leq i_{j-1}$  and  $l \geq i_j$ . Thus, for  $j = 0, \dots, k$ ,

$$\begin{aligned} e_{i_{j-1}}^T V &= d_{i_{j-1}} e_{i_{j-1}}^T (\mathbf{1} - C_k)^{-1} \cdots (\mathbf{1} - C_1)^{-1} = d_{i_{j-1}} e_{i_{j-1}}^T (\mathbf{1} - C_j)^{-1} \pmod{\mathfrak{L}_{i_{j-1}}} \\ &= d_{i_{j-1}} c_{i_{j-1}} \cdots c_{i_j} e_{i_j}^T \pmod{\mathfrak{L}_{i_{j-1}}}. \end{aligned}$$

Similar argument shows that  $e_l^T V \in \mathfrak{L}_{i_{j-1}}$  for  $l < i_{j-1}$ . This implies

$$\begin{aligned} e_1^T V^j &= e_1^T V^j \pmod{\mathfrak{L}_{i_{j-1}}} = \left( \prod_{\beta=0}^{j-1} d_{i_\beta} c_{i_\beta} \cdots c_{i_{\beta+1}-1} \right) e_{i_j} \pmod{\mathfrak{L}_{i_{j-1}}} \\ &= \left( c_1 d_1 \prod_{\beta=2}^{i_j-1} c_\beta d_\beta^{1-\epsilon_\beta} \right) e_{i_j}^T \pmod{\mathfrak{L}_{i_{j-1}}}. \end{aligned} \quad (3.9)$$

On the other hand, for  $l \in \{i_{j-1}, \dots, i_j - 1\}$ , define  $m \geq 0$  to be a number such that  $l + \beta = i_{j+\beta-1}$  for  $\beta = 1, \dots, m$  and  $l + m + 1 < i_{j+m}$ . In other words,  $l + m + 1$  is the smallest index greater than  $l$  that does not belong to the index set  $I$ . Then

$$e_l^T V^{-1} = e_l^T (\mathbf{1} - C_j) \cdots (\mathbf{1} - C_k) D^{-1} = ((-1)^{m+1} c_l \cdots c_{l+m} d_{l+m+1}^{-1}) e_{l+m+1}^T \pmod{\mathfrak{L}_{l+m}}. \quad (3.10)$$

Let us denote by  $J = \{l_1 < \dots < l_{n-k-1}\}$  the set  $\{1, \dots, n\} \setminus I$ . Then (3.10) implies

$$\begin{aligned} e_1^T X^{-\alpha} &= e_1^T V^{-\alpha} \pmod{\mathfrak{L}_{l_{\alpha-1}}} \\ &= \left( (-1)^{l_{\alpha-1}} c_1 \cdots c_{l_{\alpha-1}} d_{l_1}^{-1} \cdots d_{l_{\alpha}}^{-1} \right) e_{l_{\alpha}}^T \pmod{\mathfrak{L}_{l_{\alpha-1}}}. \end{aligned} \quad (3.11)$$

Note now that  $l_{\alpha} = i$  if and only if  $\alpha = \sum_{\beta=1}^{i-1} \epsilon_{\beta}$ . Furthermore,  $d_{l_1}^{-1} \cdots d_{l_{\alpha}}^{-1} = \prod_{\beta=2}^i d_{\beta}^{-\epsilon_{\beta}}$ . Thus, one can re-write (3.11) as

$$e_1^T V^{-\sum_{\beta=1}^{i-1} \epsilon_{\beta}} = (-1)^{i-1} \prod_{\beta=1}^{i-1} c_{\beta} d_{\beta+1}^{-\epsilon_{\beta+1}} e_i^T \pmod{\mathfrak{L}_{i-1}} = (-1)^{i-1} c_1 d_1 \frac{\prod_{\beta=2}^{i-1} c_{\beta} d_{\beta}^{1-\epsilon_{\beta}}}{d_1 \cdots d_i} \pmod{\mathfrak{L}_{i-1}}. \quad (3.12)$$

Combining (3.9), (3.12) with (2.12), one concludes that

$$e_1^T X^{\nu_i} = \gamma_i e_i^T \pmod{\mathfrak{L}_{i-1}}, \quad (3.13)$$

where  $\gamma_i$  is defined by (3.8). This implies the statement of the lemma.  $\square$

### Corollary 3.3

$$e_1^T X^{\alpha} p_i(X) e_1 = 0, \quad 1 - \sum_{\beta=1}^i \epsilon_{\beta} \leq \alpha \leq i - \sum_{\beta=1}^i \epsilon_{\beta} \quad (3.14)$$

**Proof.** By (3.7),  $e_1^T X^{\nu_l} e_{i+1} = 0$  for  $l = 1, \dots, i$ . But, by Lemma 2.1,  $e_{i+1} = p_i(X)e_1$ . Then (3.14) follows from (2.13).  $\square$

Define Toeplitz matrices

$$T_i^{(l)} = (s_{l+\alpha-\beta})_{\alpha,\beta=1}^i \quad (3.15)$$

and Toeplitz determinants

$$\Delta_i^{(m)} = \det T_i^{(i+m-\sum_{\beta=1}^i \epsilon_\beta)} \quad (m \in \mathbb{Z}) . \quad (3.16)$$

Let us also define polynomials

$$\mathcal{P}_i^{(l)}(\lambda) = \det \begin{bmatrix} s_l & s_{l+1} & \cdots & s_{l+i} \\ \cdots & \cdots & \cdots & \cdots \\ s_{l-i+1} & s_{l-i+2} & \cdots & s_{l+1} \\ 1 & \lambda & \cdots & \lambda^i \end{bmatrix} . \quad (3.17)$$

$\mathcal{P}_i^{(l)}$  is a determinant of the  $(i+1) \times (i+1)$  matrix obtained from  $T_{i+1}^{(l)}$  by replacing the last row with  $(1, \lambda, \dots, \lambda^i)$ . The following simple lemma is reminiscent of the construction of classical orthogonal polynomials on the real line and the unit circle (see, e.g. [1]) and will be useful for us in what follows.

**Lemma 3.4** *Let  $X$  be invertible matrix with a moment sequence (3.1). For  $m \in \mathbb{Z}$ , define a Laurent polynomial*

$$R(\lambda) = \lambda^m \mathcal{P}_i^{(l)}(\lambda) . \quad (3.18)$$

Then

$$e_1^T R(X) X^\alpha e_1 = 0 \quad (\alpha = l+1-m-i, \dots, l-m) \quad (3.19)$$

$$e_1^T R(X) X^{l-m+1} e_1 = (-1)^i \det T_{i+1}^{(l+1)} \quad (3.20)$$

$$e_1^T R(X) X^{l-m-i} e_1 = \det T_{i+1}^{(l)} . \quad (3.21)$$

Proof.  $R(\lambda)$  can be written as  $\sum_{\beta=0}^i R_\beta \lambda^{m+\beta}$ , where  $R_\beta$  is equal to  $(-1)^{i+\beta}$  times the minor obtained by deleting the last row and  $(\beta+1)$ st column in (3.17). Then

$$e_1^T R(X) X^\alpha e_1 = \sum_{\beta=0}^i R_\beta s_{m+\alpha+\beta} . \quad (3.22)$$

If  $\alpha$  is in the range specified in (3.19), the right-hand side of (3.22) becomes a determinant in which two of the rows coincide. Equalities (3.20), (3.21) are obtained in a similar way.  $\square$

Lemmas above imply the following

**Proposition 3.5** *Assume that  $\Delta_i^{(0)} \neq 0$  for  $i = 1, \dots, n$ . Then polynomials defined by formulas*

$$p_i(\lambda) = \frac{1}{\Delta_i^{(0)}} \mathcal{P}_i^{(i-\sum_{\beta=1}^i \epsilon_\beta)}(\lambda) . \quad (3.23)$$

satisfy (3.2, 3.3, 3.4).

**Proof.** If  $p_i(\lambda) = \sum_{l=0}^i p_{il}\lambda^l$  is a solution of 3.2, 3.3, 3.4), then by Corollary 3.3 and (3.1), we have

$$e_1^T X^\alpha p_i(X) e_1 = 0 \quad \left(1 - \sum_{\beta=1}^i \epsilon_\beta \leq \alpha \leq i - \sum_{\beta=1}^i \epsilon_\beta\right). \quad (3.24)$$

If  $\Delta_i^{(0)} \neq 0$ , Lemma 3.4 implies that (3.23) is a unique monic polynomial of degree  $i$  that satisfies (3.24). Therefore it has to coincide with the unique solution of (3.2). This completes the proof.  $\square$

Now we will be able to express parameters  $c_i, d_i$  in terms of Toeplitz determinants (3.16). First, we need the following elementary lemma.

**Lemma 3.6** *Let  $X$  be an arbitrary upper Hessenberg matrix and let  $D_0 = 1$  and  $D_i$  ( $i = 1, \dots, n$ ) denote the left upper  $i \times i$  principal minor of  $X$ . Then*

$$D_i = (-1)^i p_i(0), \quad (3.25)$$

where  $p_i(\lambda)$  are polynomials defined by (3.2).

**Proof.** The Laplace expansion w.r.t  $i$ th column leads to the following recursion for  $D_i$ :

$$D_i = \sum_{\alpha=1}^i (-1)^\alpha x_{\alpha i} D_{\alpha-1}$$

or

$$D_i + \sum_{\alpha=1}^i (-1)^{\alpha-1} x_{\alpha i} D_{\alpha-1} = 0,$$

which coincides with the recursion for  $p_i(0)$  obtained when one sets  $\lambda = 0$  in (3.2).  $\square$

**Theorem 3.7**

$$d_i = \frac{\Delta_i^{(1)} \Delta_{i-1}^{(0)}}{\Delta_i^{(0)} \Delta_{i-1}^{(1)}}, \quad c_i = -\frac{\Delta_{i+1}^{(\epsilon_{i+1})} \Delta_{i-1}^{(1-\epsilon_i)}}{\Delta_i^{(0)} \Delta_i^{(1)}}. \quad (3.26)$$

**Proof.** First note that, by (3.23), the right-hand side of (3.25) is equal to  $\frac{\Delta_i^{(1)}}{\Delta_i^{(0)}}$ , whereas, due to (2.5), the left-hand side is  $d_1 \cdots d_i$ . This immediately implies the first of the formulae (3.26).

Secondly, it is clear from (3.8), that parameters  $c_i$  are uniquely determined by  $d_i$  and  $\gamma_i$ . On the other hand, (3.8), together with (3.6), gives

$$\gamma_{i+1} = e_1^T X^{\nu_{i+1}} p_i(X) e_1. \quad (3.27)$$

Recall that, if  $\epsilon_{i+1} = 1$  then  $\nu_{i+1} = -\sum_{\beta=1}^i \epsilon_\beta$  and  $i - \sum_{\beta=1}^i \epsilon_\beta = i + 1 - \sum_{\beta=1}^{i+1} \epsilon_\beta$ . Otherwise,  $\nu_{i+1} = i + 1 - \sum_{\beta=1}^i \epsilon_\beta = i + 1 - \sum_{\beta=1}^{i+1} \epsilon_\beta$ . Comparing (3.27) with equalities (3.20), (3.21) and then using (3.16), one obtains

$$\gamma_{i+1} = (-1)^{i\epsilon_{i+1}} c_1 d_1 \frac{\prod_{\beta=2}^i d_\beta^{1-\epsilon_\beta} c_\beta}{(d_1 \cdots d_{i+1})^{\epsilon_{i+1}}} = (-1)^{i(1-\epsilon_{i+1})} \frac{\Delta_{i+1}^{(0)}}{\Delta_i^{(0)}} \quad (3.28)$$

Now, to finish the proof, it is enough to check that when one substitutes expressions (3.26) into the right-hand side of (3.8), the result agrees with (3.28). It is not hard to see that if  $d_\beta, c_\beta$  are defined by (3.26) then

$$d_\beta^{1-\epsilon_\beta} c_\beta = -\frac{\Delta_{\beta+1}^{(\epsilon_{\beta+1})} \Delta_{\beta-1}^{(0)}}{\Delta_\beta^{(\epsilon_\beta)} \Delta_\beta^{(0)}} , \quad (3.29)$$

while  $d_1 c_1 = -\frac{\Delta_2^{(\epsilon_2)}}{\Delta_1^{(1)}}$ . Therefore, the numerator in the right-hand side of (3.8) is equal to  $\frac{\Delta_i^{(\epsilon_i)}}{\Delta_{i-1}^{(0)}}$  and denominator is equal to  $\left(\frac{\Delta_i^{(1)}}{\Delta_i^{(0)}}\right)^{\epsilon_i}$ . For both possible values of  $\epsilon_i$ , this implies that formulae (3.8), (3.26) and (3.28) agree.  $\square$

One of the consequences of Lemma 3.2 is an existence of the unique sequence of Laurent polynomials  $r_0(\lambda), \dots, r_{n-1}(\lambda)$  of the form

$$r_i(\lambda) = \lambda^{\nu_{i+1}} + \sum_{\beta=1}^i r_{i\beta} \lambda^{\nu_\beta} . \quad (3.30)$$

such that

$$e_1^T r_{i-1}(X) = \gamma_i e_i^T \quad (i = 1, \dots, n) \quad (3.31)$$

Next proposition shows that  $r_i$  can be conveniently described by formulas similar to (3.23).

**Proposition 3.8**

$$r_i(\lambda) = \frac{(-1)^{i\epsilon_{i+1}} \lambda^{1-\sum_{\beta=1}^{i+1} \epsilon_\beta}}{\Delta_i^{(0)}} \mathcal{P}_i^{(i-\sum_{\beta=1}^{i+1} \epsilon_\beta)}(\lambda) \quad (3.32)$$

**Proof.** First note, that by (2.13), (3.30) can be re-written as

$$r_i(\lambda) = \lambda^{1-\sum_{\beta=1}^{i+1} \epsilon_\beta} \sum_{\beta=1}^i a_{i\beta} \lambda^{\nu_\beta} . \quad (3.33)$$

Next, by (2.11),  $\nu_{i+1}$  coincides with the lowest (resp. highest) degree in  $r_i(\lambda)$  if  $\epsilon_{i+1} = 1$  (resp.  $\epsilon_{i+1} = 0$ ). Finally, due to Lemma 3.1, the claim that  $e_1^T r_i(X)$  is proportional to  $e_{i+1}^T$  is equivalent to a property

$$e_1^T r_i(X) X^l e_1 = 0 \quad (l = 0, \dots, i-1) \quad (3.34)$$

This property is satisfied by Lemma 3.4.  $\square$

**Corollary 3.9** Define a bilinear form  $(,)$  on  $\mathbb{C}[\lambda, \lambda^{-1}]$  by

$$(\lambda^i, \lambda^j) = s_{i+j} \quad (i, j \in \mathbb{Z}) .$$

Then functions  $p_0(\lambda), \dots, p_{n-1}(\lambda); r_0(\lambda), \dots, r_{n-1}(\lambda)$  form a bi-orthogonal system w. r. t.  $(,)$ , i. e.

$$(r_i(\lambda), p_j(\lambda)) = \delta_i^j c_1 d_1 \frac{\prod_{\beta=2}^{i-1} d_\beta^{1-\epsilon_\beta} c_\beta}{(d_1 \dots d_i)^{\epsilon_i}} . \quad (3.35)$$

**Proof.** Follows immediately from Lemmas 3.1, 3.2 and Propositions 3.4, 3.8.  $\square$



## 4 Symmetric Case

It is natural to ask how should results of the previous section be modified, if initially one identifies  $\mathfrak{b}_-^*$  with a space  $\mathfrak{S}$  of symmetric matrices rather than with  $\mathcal{H}$ . In this case, a co-adjoint orbit of  $\mathbf{B}_-$  through  $X_0 \in \mathfrak{S}$  is described as

$$\mathfrak{O}_{X_0} = \{(\text{Ad}_n X_0)_{sym} : n \in \mathbf{B}_-\} , \quad (4.1)$$

where  $A_{sym} = A_{\geq 0} + (A_{>0})^T$ .

A set  $M_I$  is still defined by (2.3, but the definition of  $X_I$  should be changed as follows:

$$X_I = e_{1i_1} + e_{i_1 1} + \sum_{j=1}^{k-1} (e_{i_j i_{j+1}} + e_{i_{j+1} i_j}) . \quad (4.2)$$

An open dense subset  $M'_I \subset M_I$ , that we are going to study, consists of elements of the form

$$X = (\mathbf{1} - U_2^T)^{-1} (\mathbf{1} + U_1^T) D (\mathbf{1} + U_1) (\mathbf{1} - U_2)^{-1}, \quad (4.3)$$

where, as before,  $D = \text{diag}(d_1, \dots, d_2)$  and  $U_1, U_2$  are defined by (2.10). Matrix entries of  $X$  then are found to be

$$x_{lm} = x_{ml} = \begin{cases} c_l \cdots c_{m-1} (d_l + \sum_{\alpha=i_{j-1}}^{l-1} d_\alpha (c_\alpha \cdots c_{l-1})^2), & i_{j-1} < l \leq m \leq i_j \\ d_{i_{j-1}} c_{i_{j-1}} \cdots c_{m-1}, & i_{j-1} = l < m \leq i_j \\ 0, & \text{otherwise} \end{cases} \quad (4.4)$$

A similar expression can be obtained for matrix entries of  $X^{-1}$ . Denote by  $J$  a set of indices  $(\{1, \dots, n\} \setminus I) \cup \{n\} = \{l_0 = 1 < l_1 < \dots < l_{n-k} = n\}$ . Then

$$(X^{-1})_{lm} = (X^{-1})_{ml} = \begin{cases} (-1)^{m-l} c_l \cdots c_{m-1} \left( \frac{1}{d_m} + \sum_{\alpha=m}^{l_j-1} \frac{1}{d_{\alpha+1}} (c_m \cdots c_\alpha)^2 \right), & l_{j-1} \leq l \leq m < l_j \\ (-1)^{m-l} c_l \cdots c_{l_j-1} \frac{1}{d_{l_j}}, & l_{j-1} = l < m = l_j \\ 0, & \text{otherwise} \end{cases} \quad (4.5)$$

Note that a conjugation of  $X$  by the matrix  $\text{diag}(\mathbf{1}_i, -\mathbf{1}_{n-i})$  does not change values of parameters  $d_j, j = 1, \dots, n$  and  $c_j, j \neq i$ , but changes  $c_i$  to  $-c_i$ . This conjugation also does not affect values of the moments of  $X$ . Thus, in order to make a solution of the inverse problem below unique, we shall assume that all  $c_i$  are positive.

As in the previous section, we are interested in expressing parameters  $c_i, d_i$  via the moment sequence  $(s_i = s_i(X) = e_1^T X^i e_1)_{i \in \mathbb{Z}}$  of  $X$ .

We start by noting that Lemma 3.2 remains literally true for matrices  $X$  of the form 4.3, which implies an existence of the unique sequence of Laurent polynomials  $r_0(\lambda), \dots, r_{n-1}(\lambda)$  of the form (3.30) satisfying (3.31). Formulas for functions  $r_i(\lambda)$  are similar to (3.32).

First, we define a new collection of Toeplitz determinants

$$\mathfrak{D}_i^{(m)} = \det T_i^{(i+1+m-2 \sum_{\beta=1}^i \epsilon_\beta)} \quad (m \in \mathbb{Z}) . \quad (4.6)$$

**Proposition 4.1**

$$r_{i-1}(\lambda) = \frac{(-1)^{(i-1)\epsilon_i} \lambda^{1-\sum_{\beta=1}^i \epsilon_\beta}}{\mathfrak{D}_{i-1}^{(0)}} \mathcal{P}_{i-1}^{(i-2\sum_{\beta=1}^{i-1} \epsilon_\beta - \epsilon_i)}(\lambda) . \quad (4.7)$$

**Proof.** We can argue exactly as in the proof of Proposition 3.8. Note, however, that since  $X$  is symmetric,  $r_{i-1}(X)e_1 = \gamma_i e_i$ . This means that condition (3.34) that guarantees (3.31) has to be replaced by

$$e_1^T r_{i-1}(X) X^{\nu_\alpha} e_1 = 0 \quad (\alpha = 1, \dots, i-1)$$

or, equivalently,

$$e_1^T r_{i-1}(X) X^l e_1 = 0 \quad (l = 1 - \sum_{\beta=1}^{i-1} \epsilon_\beta, \dots, i-1 - \sum_{\beta=1}^{i-1} \epsilon_\beta) .$$

Function  $r_{i-1}$  defined by (4.7) satisfies this condition for exactly the same reason that function (3.32) satisfies (3.34).  $\square$

**Corollary 4.2**

$$\gamma_i^2 = (-1)^{i-1} \frac{\mathfrak{D}_i^{(0)}}{\mathfrak{D}_{i-1}^{(0)}} \quad (4.8)$$

**Proof.** It follows from (3.7), (3.31) and symmetricity of  $X$  that

$$\gamma_i^2 = e_1^T r_{i-1}(X) X^{\nu_i} e_1 . \quad (4.9)$$

Then (4.8) follows from Lemma 3.4, (2.11) and (4.7).  $\square$

**Remark.** Note that, as one would expect, in the case of the classical moment problem ( $I = \{2, 3, \dots, n\}$ ,  $X$  is tri-diagonal), a condition that ensures that the right-hand side of (4.8) is positive is a positive definiteness of the Hankel matrix  $(s_{i+j-2})_{i,j=1}^n$ . Similarly, in the case  $I = \{n\}$  which relevant in the study of the peakons lattice and was comprehensively studied in [2], one comes to a conclusion that the Hankel matrix  $(s_{2-i-j})_{i,j=1}^n$  is positive definite.

Let us consider now an  $m \times m$  sub-matrix  $X_m$  of  $X \in M'_l$  obtained by deleting  $(n-m)$  last rows and columns. It is clear from (4.4), that  $X_m$  does not depend on parameters  $c_m, \dots, c_{n-1}, d_{m+1}, \dots, d_n$ .

**Lemma 4.3** *Let  $s_\alpha(X_m) = e_1 X^\alpha e_1^T$ . then*

$$s_\alpha(X_m) = s_\alpha(X) \quad (2 - 2 \sum_{\beta=1}^m \epsilon_\beta \leq \alpha \leq 2m + 1 - 2 \sum_{\beta=1}^m \epsilon_\beta) . \quad (4.10)$$

**Proof.** For  $l > 0$ , an expression for  $s_l(X)$  in terms of matrix entries of  $X$  reads  $s_l(X) = \sum_{\alpha_1, \dots, \alpha_{l-1}} x_{1\alpha_1} x_{\alpha_1\alpha_2} \cdots x_{\alpha_{l-1}1}$ . By (4.4), many of the terms in this sum are zero. Moreover, if

$l < 2k + 1$ , where  $k$  is the cardinality of  $I$ , one can find among the non-zero terms the one where  $\max(\alpha_1, \dots, \alpha_{l-1})$  reaches its maximum. This term is equal to  $\prod_{\beta=1}^q x_{i_{\beta-1}, i_{\beta}}^2$ , if  $l = 2q$  and

$x_{i_q i_q} \prod_{\beta=1}^q x_{i_{\beta-1}, i_{\beta}}^2$ , if  $l = 2q + 1$ . This implies that if  $m > i_j$  and  $0 < l \leq 2j + 1$  then the expression for  $s_l(X)$  involves only entries of  $X_m$  and, therefore,  $s_l(X) = s_l(X_m)$ . Since the largest  $j$  such that  $m > i_j$  is given by  $m - \sum_{\beta=1}^m \epsilon_{\beta}$ , (4.10) is satisfied for  $\alpha = 0, \dots, 2m + 1 - 2 \sum_{\beta=1}^m \epsilon_{\beta}$ .

Similarly, one can use (4.5) to conclude that if  $q$  is the largest index such that  $l_q < m$  then (i) for  $\alpha, \beta \leq l_q$ ,  $(X_m^{-1})_{\alpha\beta} = (X^{-1})_{\alpha\beta}$ ; (ii) for  $-2q \leq l < 0$ , the expression for  $s_l(X)$  contains only entries  $(X^{-1})_{\alpha\beta}$  with  $\alpha, \beta \leq l_q$ , which means that in this case  $s_l(X)$  coincides with  $s_l(X_m)$ . To finish the proof, it remains to notice that  $q$  is equal to  $\sum_{\beta=1}^m \epsilon_{\beta} - 1$ .  $\square$

#### Proposition 4.4

$$\det(\lambda - X_m) = \frac{1}{\mathfrak{D}_m^{(0)}} \mathcal{P}_{m+1}^{(m+1-2 \sum_{\beta=1}^m \epsilon_{\beta})}(\lambda) . \quad (4.11)$$

**Proof.** Let  $\lambda^m + \sum_{i=0}^{m-1} a_{mi} \lambda^i$ . Then the Hamilton-Cayley theorem implies

$$s_{\alpha+m}(X_m) + \sum_{i=0}^{m-1} a_{mi} s_{\alpha+i}(X_m) = 0 \quad (\alpha \in \mathbb{Z}) . \quad (4.12)$$

By Lemma 4.3, (4.12) remains valid if we replace  $s_{\alpha+i}(X_m)$  with  $s_{\alpha+i} = s_{\alpha+i}(X)$  for  $i = 0, \dots, m$ , as long as  $2 - 2 \sum_{\beta=1}^m \epsilon_{\beta} \leq \alpha \leq m + 1 - 2 \sum_{\beta=1}^m \epsilon_{\beta}$ . This means that, after the right multiplication of the matrix used in the definition (3.17) of  $\mathcal{P}_{m+1}^{(m+1-2 \sum_{\beta=1}^m \epsilon_{\beta})}$  by the unipotent matrix  $(\mathbf{1} + \sum_{\beta=1}^{m-1} e_{\beta+1, m+1})$ , one gets a matrix of the form

$$\begin{bmatrix} T_m^{(m+1-2 \sum_{\beta=1}^m \epsilon_{\beta})} & 0 \\ 1 \ \lambda \ \dots \ \lambda^{m-1} & \det(\lambda - X_m) \end{bmatrix}$$

and (4.11) follows.  $\square$

It drops out immediately from (4.11) and (4.3) that

$$d_1 \cdots d_m = \frac{\mathfrak{D}_m^{(1)}}{\mathfrak{D}_m^{(0)}} . \quad (4.13)$$

Now the analogue of Theorem 3.7 for matrices (4.3) drops out immediately from (4.13), (4.8) and (3.8).

#### Theorem 4.5

$$d_i = \frac{\mathfrak{D}_i^{(1)} \mathfrak{D}_{i-1}^{(0)}}{\mathfrak{D}_i^{(0)} \mathfrak{D}_{i-1}^{(0)}} , \quad c_i = \frac{\left( -\mathfrak{D}_{i+1}^{(0)} \mathfrak{D}_{i-1}^{(1)} \right)^{\frac{1}{2}}}{\mathfrak{D}_i^{(1)}} \left( \frac{\mathfrak{D}_{i+1}^{(1)}}{\mathfrak{D}_{i+1}^{(0)}} \right)^{\epsilon_{i+1}} \left( \frac{\mathfrak{D}_{i-1}^{(0)}}{\mathfrak{D}_{i-1}^{(1)}} \right)^{1-\epsilon_i} . \quad (4.14)$$

## 5 Conclusion

The results of this paper can be used to make several constructions used in the study of integrable lattices of Toda type more explicit. For example, in [8] we proved that for any  $I, J$  there exists a bi-rational Poisson map from  $M'_I$  to  $M'_J$ , that intertwines the Toda flows on  $M'_I$  and  $M'_J$ . A construction we gave was by induction and expressions for matrix entries of elements of  $M'_J$  in terms of matrix entries of elements of  $M'_I$  were very involved. Now we can give explicit formulas for this map using just (3.1) and Theorem 3.7. Similarly, Theorem 4.5 can be used to simplify (in the case of elementary orbits) the construction of the Poisson map from the Kostant-Toda to symmetric Toda flows proposed in [4].

A solution to the inverse problem can also be useful in avoiding blow-ups of the Toda flows by switching from one elementary orbit to another, which, in turn, may become important in the context of  $LU$  type algorithms for computing eigenvalues of Hessenberg matrices. A natural question that arises in this connection is to give an intrinsic description of all sequences  $(s_i)$  from which an element of  $M'_I$  can be restored for at least one  $I$ .

Because of possible implications in the coding theory, it would be also interesting to study solvability of the inverse problem over a finite field (cf. [7] where the connection with the coding theory was observed in the tri-diagonal case).

Another direction of possible investigation that we plan to pursue in the future is an extension of our results from  $M'_I$  to  $M_I$  (to this end, some of the genericity assumptions will have to be relaxed) and then to more general co-adjoint orbits. In the latter case, one has to restore an element of the orbit from a collection of rational functions. In the case of generic co-adjoint orbits this has been done in [6, 12].

## References

- [1] N. I. Akhiezer, The classical moment problem and some related questions in analysis. Hafner Publishing Co., New York 1965.
- [2] R. Beals, D. H. Sattinger and J. Szmigielski [1999], Multipeakons and the classical moment problem, *Adv. Math.* **154** (2000), 229–257.
- [3] Yu. M. Berezansky, The integration of semi-infinite Toda chain by means of inverse spectral problem, *Rep. Math. Phys.* **24** (1986), 21–47.
- [4] A. M. Bloch, M. Gekhtman, Hamiltonian and gradient structures in the Toda flows, *Journ. of Geom. & Phys.* **27**, 230–248 (1998)
- [5] R. Brockett, L. Faybusovich, Toda flows, inverse spectral transform and realization theory. *Systems Control Lett.* **16** (1991), 79–88.
- [6] P. Deift, L. Li, T. Nanda and C. Tomei, The Toda flow on a generic orbit is integrable, *Comm. Pure & Appl. Math.* **39** (1986) 183–232.
- [7] L. Faybusovich, On the Rutishauser’s approach to eigenvalue problems, In: Linear algebra for control theory, 87–102, *IMA Vol. Math. Appl.* **62** (1994), 87–102.
- [8] L. Faybusovich, M. I. Gekhtman, Elementary Toda orbits and integrable lattices, *J. Math. Phys.* **41** (2000), 2905–2921.

- [9] L. Faybusovich, M. I. Gekhtman, Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices, *Phys. Lett. A* **272** (2000), 236–244.
- [10] S. Kharchev, A. Mironov, A. Zhedanov, Faces of relativistic Toda chain, *Int. J. Mod. Phys.* **12** (1997), 2675–2724.
- [11] J. Moser, Finitely many mass points on the line under the influence of an exponential potential. *Batelles Recontres, Springer Notes in Physics* (1974), 417–497.
- [12] S. Singer, Doctoral Dissertation. Courant Institute of Mathematical Sciences. New York University (1990)
- [13] Y. B. Suris, Integrable discretizations for lattice systems: local equations of motion and their hamiltonian properties, *Rev. Math. Phys.* **11** (1999), 727–822.
- [14] Y. B. Suris, Private communication.